Lecture notes on Angular Momentum and Tensor Operators

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Preface

These lecture notes are meant as an overview and introduction to angular momentum and tensor operators. These pages are to a large extent based on the text by Lindgren & Karlsson “Tensor operatorer med vinkelkorrelationsteori” but with several additions and clarifications. The notes should be used as a complement to standard texts such as Edmonds “Angular Momentum in Quantum Mechanics”, Lindgren & Karlsson “Tensor operatorer med vinkelkorrelationsteori”, Zare “Angular Momentum”, Judd “Operator Techniques in Atomic Spectroscopy”, or Condon & Shortley “The Theory of Atomic Spectra”, mentioning only a few.

Contents

Chapter 1

Angular momentum

Why would angular momentum interest anyone? Well, without a solid theory of angular momentum we would have severe difficulties in solving central force problems, *e.g.* the hydrogen atom and problems involving the central field approximation, actually most problems with spherical symmetry. This should be enough to motivate most of us. For those that still are not convinced that "refreshing" the knowledge in angular momentum theory is worthwhile, one could add that the angular momentum concept also play an important role for modern nuclear- and particle physics theory...

1.1 Definitions

In classical mechanics the angular momentum $\bar{L}$ for n massive particles is defined as

$$\bar{L} = \sum_{i=1}^{n} \bar{r}_i \times \bar{p}_i \tag{1.1}$$

where $\bar{r}_i$ is the position vector and $\bar{p}_i$ is the linear momentum vector, see figure **??**.

fig1.

For a system without external torques all three components of $\bar{L}$ are constants of motion. In a historic paper 1913, Bohr postulated that the angular momentum $\bar{L}$ of a system was quantized, and in 1916 Sommerfeld suggested that for an electron in a closed orbit both direction and magnitude was quantized.

In quantum mechanics the angular momentum operator is defined as

$$\bar{L} = -i\hbar\bar{r} \times \bar{\nabla} \qquad (\bar{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}) \tag{1.2}$$

which is in complete analogy with Eq.(1.1). The following commutation relations apply:

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k \qquad (i, j, k = x, y, z) \tag{1.3}$$

where ϵ_{ijk} is the Levi-Civita (or permutation) symbol. The commutation relation is easily proven

$$\begin{aligned}
[L_i, L_j] &= -\hbar^2\{(\epsilon_{ijk} r_j \nabla_k \epsilon_{jki} r_k \nabla_i) - (\epsilon_{jki} r_k \nabla_i \epsilon_{ijk} r_j \nabla_k)\} \\
&= -\hbar^2\{\epsilon_{ijk}\epsilon_{ijk}(r_j \nabla_i + r_j r_k \nabla_k \nabla_i) - \epsilon_{ijk}\epsilon_{ijk}(r_k \delta_{ij} \nabla_k + r_k r_j \nabla_i \nabla_k)\} \\
&= [\epsilon_{iik} = 0] = -\hbar^2 \epsilon_{ijk}\epsilon_{kij} r_j \nabla_i = \hbar^2 \epsilon_{ijk}\epsilon_{kji} r_j \nabla_i = -i\hbar^2 \epsilon_{ijk}(\bar{r} \times \bar{\nabla})_k \\
&= i\hbar \epsilon_{ijk} L_k
\end{aligned}$$

In the general theory of angular momentum the commutation relation Eq. (1.3) is used as definition, not Eq. (1.2)

$$[J_i, J_j] = i\hbar\epsilon_{ijk} J_k \tag{1.4}$$

The square of the total angular momentum J^2 commutes with all components of J,

$$[J^2, J_i] = 0 \qquad (i = x, y, z) \tag{1.5}$$

$$\begin{aligned}
[J^2, J_z] &= [J_x^2 + J_y^2 + J_z^2, J_z] \\
&= J_x[J_x, J_z] + [J_x, J_z]J_x + J_y[J_y, J_z] + [J_y, J_z]J_y + J_z[J_z, J_z] + [J_z, J_z]J_z \\
&= J_x(-i\hbar J_y) + (-i\hbar J_y J_x) + J_y(i\hbar J_x) + (i\hbar J_x J_y) + 0 + 0 \\
&= 0
\end{aligned}$$

Associated with J are the kets $|JM\rangle$ for which we may write

$$J_z|JM\rangle = M\hbar|JM\rangle \tag{1.6}$$

$$J^2|JM\rangle = J(J+1)\hbar^2|JM\rangle \tag{1.7}$$

J and M are quantum numbers restricted to either half-integer or integer values. We will now prove Eq. (1.7) and show that for a given J, the allowed values of M are $M = -J, -J+1, \ldots, J$. It turns out useful to introduce the ladder operators

$$J_\pm = J_x \pm iJ_y \tag{1.8}$$

Using the commutation relations for the components of J (Eq. (1.4)) it is straight forward to show

$$[J_+, J_-] = 2\hbar J_z \tag{1.9}$$

$$[J_z, J_\pm] = \pm\hbar J_\pm \tag{1.10}$$

$$[J^2, J_\pm] = 0 \tag{1.11}$$

Eq. (1.11) is obviously a direct consequence of Eq. (1.5). The effect of $J_\pm$ is to increase/decrease M with one unit $\hbar$, leaving J unchanged. To see this we look at the following expression.

$$\begin{aligned} J_z(J_\pm|JM\rangle) &= ([J_z, J_\pm] + J_\pm J_z)|JM\rangle = (\pm\hbar J_\pm + J_\pm M\hbar)|JM\rangle \\ &= (M\hbar \pm \hbar)J_\pm|JM\rangle \end{aligned} \tag{1.12}$$

and we see that $J_\pm$ increase/decrease the J_z eigenvalue with $\hbar$. The same calculation for J^2 give

$$J^2(J_\pm|JM\rangle) = J_\pm J^2|JM\rangle = J(J+1)\hbar^2 J_\pm|JM\rangle \tag{1.13}$$

because $[J^2, J_\pm] = 0$.

Returning to the eigenvalue problem of J^2 and J_z. Suppose we have an upper limit on M namely M_{max}. In such a case

$$J_+|JM_{max}\rangle = 0 \tag{1.14}$$

Because $[J_+, J_-] \neq 0$, we also see that

$$J_-J_+|JM_{max}\rangle = 0 \tag{1.15}$$

With Eqs. (1.8) and (1.4) it is found that

$$\begin{aligned} J_-J_+|JM_{max}\rangle &= (J_x^2 + J_y^2 - i(J_yJ_x - J_xJ_y))|JM_{max}\rangle \\ &= (J^2 - J_z^2 - \hbar J_z)|JM_{max}\rangle \end{aligned}$$

and therefore $(a - M_{max}^2\hbar^2 - \hbar M_{max}\hbar) = 0$, where a denote the eigenvalue of J^2. This expression follow because the ket is not the null ket. Continuing the massage of our expressions,

$$a = M_{max}\hbar(M_{max}\hbar + \hbar) \tag{1.16}$$

In analogy with Eq. (1.14) we have

$$J_-|JM_{min}\rangle = 0 \tag{1.17}$$

and therefore

$$a = M_{min}\hbar(M_{min}\hbar - \hbar) \tag{1.18}$$

Eqs. (1.16) and (1.18) result in

$$M_{min} = -M_{max} \tag{1.19}$$

Using the ladder operator we can now get from $|JM_{min}\rangle$ to $|JM_{max}\rangle$ (apply J_+ n times) and

$$M_{max}\hbar = M_{min}\hbar + n\hbar \tag{1.20}$$

Eq. (1.19) then give

$$M_{max} = \frac{n}{2} \tag{1.21}$$

Finally, by setting $J = n/2$

$$M = -J, -J+1, \ldots, J \tag{1.22}$$

so from Eq. (1.16) it is seen that

$$a = M_{max}\hbar(M_{max}\hbar + \hbar) = J(J+1)\hbar^2 \tag{1.23}$$

To conclude we have shown that

$$\left.\begin{array}{rcl} J^2|JM\rangle & = & J(J+1)\hbar^2|JM\rangle \\ J_z|JM\rangle & = & M\hbar|JM\rangle \\ M & = & -J, -J+1, \ldots, J \end{array}\right\} \tag{1.24}$$

1.2 Spherical harmonics

The functions that satisfy Eqs. (1.6)-(1.7) are the spherical harmonics. These are defined by

$$Y_{lm}(\theta\varphi) = (-1)^m \left(\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right)^{1/2} P_l^m(\cos(\theta))\exp(im\varphi) \tag{1.25}$$

where $P_l^m(\mu)$ are the associated Legendre polynomials. These last functions are defined by

$$P_l^m(\mu) = \frac{(1-\mu)^{m/2}}{2^l l!}\frac{d^{l+m}}{d\mu^{l+m}}(\mu^2-1)^l \qquad (|\mu|\langle 1) \tag{1.26}$$

For more details the reader is referred to any standard textbook on mathematical methods in physics, *e.g.* Wyld [1] or Arfken [2]. It will turn out useful (when working with tensor operators) to introduce the following functions (Racah [3])

$$C_q^{(k)} = \left(\frac{4\pi}{2k+1}\right)^{1/2} Y_{kq} \tag{1.27}$$

Note that the letter pairs $(kq), (lm)$ and (tp) probably will be used and confused through out the course.

1.3 Quantization of angular momentum; What does it mean?

Quantization of the angular momentum in a system imply that the angular momentum components do not commute,

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$$

The uncertainty principle therefore makes it impossible to measure any two components simultaneously.

The rule for the minimum uncertainties of measurement of ΔA and ΔB of any two non-commuting operators A and B is given by

$$\overline{(\Delta A)^2(\Delta B)^2} \geq \overline{\left\{\frac{i}{2}[A, B]\right\}}^2 \tag{1.28}$$

where the bars of course imply expectation values.
Example:

By measuring J_z, obtaining $\hbar m$, we get according Eq. (1.28) for the simultaneously uncertainties of J_x and J_y

$$\overline{(\Delta J_x)^2(\Delta J_y)^2} \quad \geq \quad \overline{\left\{\frac{i}{2}[J_x, J_y]\right\}}^2 = \overline{\left\{\frac{i}{2}i\hbar J_z\right\}}^2 = \frac{\hbar^2}{4}\overline{J_z^2} = \frac{m^2\hbar^4}{4}$$

Other interesting features of the quantization is the fact that the measured values of the total angular momentum and of its component in a given direction can only take certain values, namely $\hbar^2 j(j+1)$ and $\hbar m$ $(m = -j, -j+1, \ldots, j)$. $\hbar^2 j(j+1)$ was as a matter of fact first discovered empirically.